Fractional Differential Equations with Variable Coefficients: Constructive Series Solutions

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Outline of the talk

Linear differential equation of first order

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or what is the same

$$a_1(t)D_t^1 y(t) + a_0(t)y(t) = h(t),$$

where $a_1(t) \neq 0$ for any $t \in I$, and $a_1(t)$, $a_0(t)$, $h(t)$ are continuous functions in I.

Solution

The solution of the latter equation is given by:

$$y(t)e^{\int p(t)dt} = \int e^{\int p(t)dt} Q(t)dt + C,$$

where

$$p(t) = \frac{a_0(t)}{a_1(t)} \quad \text{and} \quad Q(t) = \frac{h(t)}{a_1(t)}.$$

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Let us see an example !

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1. Find the solution of the following DE:

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with $p(\nu)=-2/\nu$ and $Q(\nu)=-6/\nu^2$. The solution is given by

$$\frac{1}{\nu^2}\mu=\int\frac{1}{\nu^2}\left(-\frac{6}{\nu^2}\right)d\nu+C \Longrightarrow \mu=\frac{2}{\nu}+C\nu^2.$$

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Next, the general case !

General differential equation with variable coefficients

We consider the following equation:

$$D_t^m y(t) + \sum_{i=1}^{N} a_i(t) D_t^{m_i} y(t) = h(t),$$

where $a_i's$ $(i = 1, \ldots, N)$ and h are continuous functions in I.

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One can find a series solution of the above equation by using the Peano–Baker series method[1].

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What to investigate now? What happens in the case of integrable coefficients? Is the Peano-Baker method good-enough? Can we find some applications of these results? etc ....

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Some questions with respect to operators

What happens if we consider some other operators that somehow extend and generalize the classical derivatives (related with the so-called Fractional calculus)? E.g.

$$D_t^2 y(t) + a_1(t) D_t^{3/2} y(t) = h(t),$$

or

$$D_t^4 y(t) + a_1(t) D_t^{5/2} y(t) + a_2(t) D_t^{3/2} y(t) + a_3(t) D_t^1 y(t) = h(t).$$

What is fractional calculus?

The underlying idea is to generalize the order of differentiation and integration outside just the set of whole numbers. We know that:

$$f(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = f'(x), \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) = f''(x),$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}f(x) = f'''(x) = f^{(3)}(x), \quad \frac{\mathrm{d}^4}{\mathrm{d}x^4}f(x) = f^{(4)}(x), \quad \dots$$

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Also the Fundamental Theorem of Calculus tells us:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_c^x f(y)\mathrm{d}y = f(x) \Longleftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}f(x) = f(x).$$

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So

$$\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}f(x) = \int_c^x f(y)\,\mathrm{d}y, \quad \frac{\mathrm{d}^{-2}}{\mathrm{d}x^{-2}}f(x) = \int_c^x \int_c^y f(z)\,\mathrm{d}z\,\mathrm{d}y,$$

$$\frac{\mathrm{d}^{-3}}{\mathrm{d}x^{-3}}f(x) = \int_c^x \int_c^y \int_c^z f(w)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}y, \quad \ldots$$

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In this way we can find the derivatives of a function $f(x)$ to any integer order:

1. Positive integers given by differentiation.
2. Negative integers by integration.
3. The 0th derivative of a function is the function itself.

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Fractional calculus asks the question: what about non-integer orders of differentiation and integration? Can we take, for example,

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}f(x) = ?? \quad \frac{\mathrm{d}^{-\pi}}{\mathrm{d}x^{-\pi}}f(x) = ?? \quad \frac{\mathrm{d}^{2+i}}{\mathrm{d}x^{2+i}}f(x) = ??$$

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We'll answer it in the next slides !

Why study fractional calculus?

It's about generalizing a very basic concept (differentiation / integration) to a new level, and generalization is what drives a lot of mathematics.

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Fractional calculus is widely studied not only by pure mathematicians but also by physicists, biologists, chemists, and engineers, because of its applications in many parts of science.

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E.g.:

- Fractional derivatives are **non-local** (as we'll see later), meaning that they depend on the behaviour of $f(x)$ not only near one specific value of x but on a larger interval. So they're useful in modelling many real-life processes with non-local effects.

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- Fractional calculus has connections with the **theory of fractals** (shapes with non-integer dimension), which are used in **chaos theory**.
- Viscoelastic substances, behaving partly like (viscous) fluids and partly like (elastic) solids, therefore have natural "fractional" properties and may best be modelled using fractional calculus.

History

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The inventor of calculus, Gottfried Leibniz, was asked about it in an exchange of letters with L'Hôpital. In a letter dated 30 September 1695, Leibniz said that taking the order of differentiation to be $\frac{1}{2}$ produced a paradox, but:

> *"It appears that one day useful consequences will be drawn from these paradoxes."*

Lacroix's idea

In 1819, Lacroix spent two pages in his 700-page treatise on calculus investigating fractional derivatives of polynomials. Generalising from the standard identity

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n},$$

Lacroix proposed the following definition for fractional derivatives of power functions:

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-\nu+1)}x^{m-\nu}, \quad \Gamma(m+1) = m!.$$

Abel's application

The first application of fractional calculus was discovered by Abel in 1823. He was studying the **tautochrone problem** – the problem of finding a curve such that an object sliding down the curve by gravity will always, regardless of its starting point, take the same amount of time to reach the bottom – and discovered an equation

$$k = \sqrt{\pi} \int_0^x (x-s)^{-1/2} f(s) \mathrm{d}s \Longleftrightarrow k = \sqrt{\pi} \frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}} f(x).$$

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The natural solution of this equation, if we assume fractional calculus works roughly as normal calculus and follows Lacroix's formula, should be

$$f(x) = \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} \left(\frac{k}{\sqrt{\pi}} \right) = \frac{k}{\pi \sqrt{x}},$$

which does indeed solve the tautochrone problem.

A deep study of FC

The first major study of fractional calculus was by Liouville in the 1830s. He defined two formulae for fractional derivatives, one based on exponential functions and one based on power functions:

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}(e^{ax}) = a^\nu e^{ax};$$
$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}(x^{-a}) = (-1)^\nu \frac{\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu}.$$

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Liouville's second formula can be deduced from Liouville's first formula. However, both of these are inconsistent with Lacroix's formula. We see the first problem in fractional calculus! Both Lacroix and Liouville seem correct, the natural generalisations of the standard repeated derivatives of power functions and exponential functions respectively, but we cannot have both of them.

Controversy and prediction

In the mid-19th century, therefore, some people followed Lacroix's formula and others followed Liouville's. Augustus de Morgan, however, predicted in 1842 that:

> *"Both these systems, then, may very possibly be parts of a more general system."*

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And indeed, in the late 19th century, inspired by a complex analysis approach discovered by Sonin in 1869, a unifying definition emerged. This is now called the **Riemann–Liouville** fractional integral, and both Lacroix's formula and Liouville's two formulae turn out to be special cases of it, proving de Morgan's words correct.

Riemann-Liouville FC and others

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Other mathematicians who touched on the concepts of fractional calculus at least briefly include Euler, Fourier, Riemann, Heaviside, Hardy & Littlewood. Many of the great names of mathematics have been associated somehow with fractional calculus, but the topic in general remained obscure until the late 20th and early 21st centuries.

RL Fractional integral

For an n-fold integral, there is a well known **Cauchy's formula for repeated integration**

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-2}} \int_a^{x_{n-1}} \varphi(x_n) \mathrm{d}x_n \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} \varphi(t) \mathrm{d}t.$$

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Hence it was formulated the **Riemann-Liouville (RL) fractional integral**

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad \alpha > 0.$$

Fractional integrals and derivatives of high order

Many different ways of extending the concept of differentiation and integration. Most common, **Riemann–Liouville** and **Caputo-Djrbashian**:

- Fractional integral ${}^{RL}_{a}I^{\alpha}_{t}f(t) := \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t (t-\tau)^{\alpha-1}f(\tau)\,\mathrm{d}\tau,$ $\alpha > 0.$
- Fractional derivative ${}^{RL}_{a}D^{\alpha}_{t}f(t) := \dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}^{RL}_{a}I^{n-\alpha}_{t}f(t)\right)$, i.e.

$$ {}^{RL}_{a}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)\,\mathrm{d}\tau,\ n := \lfloor\alpha\rfloor+1. $$

- Fractional derivative ${}^{C}_{a}D^{\alpha}_{t}f(t) := {}^{RL}_{a}I^{n-\alpha}_{t}\left(\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)\right).$

Some examples

1. If $f(t) = Constant$ then

$$ {}^{C}_{a}D^{\alpha}_{t}f(t) = 0, \quad {}^{RL}_{a}D^{\alpha}_{t}f(t) = C\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1. $$

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2. If $\beta, \alpha > 0$ then

$$ {}^{RL}_{a}I^{\alpha}_{t}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}, $$

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$$ {}^{C}_{a}D^{\alpha}_{t}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-1}, \quad \beta > m, $$

where $m = \lfloor \alpha \rfloor + 1$ for $\alpha \notin \mathbb{N}_0$, $m = \alpha$ for $\alpha \in \mathbb{N}_0$.

Some classical books on the field

Some sources:

1. S.G. Samko, A.A. Kilbas, O.I. Marichev. Fractional Integrals and Derivatives: Theory and Applications, Gordon & Breach Science Publishers, Yverdon, 1993.

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Another introductory textbook:

3. K.S. Miller, B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

Fractional differential equations: constant coefficients

Fractional differential equations (FDEs) relate a function to its fractional derivatives. For example:

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FDEs with constant coefficients can be solved by various methods. Laplace transforms, series solutions, operational calculus, exponential-type substitutions, ...

Fractional differential equations: variable coefficients

Linear inhomogeneous FDE with continuous variable coefficients:

$$\begin{cases} {}_0^C D_t^{\alpha_0} y(t) + \sum_{i=1}^{m} a_i(t)\, {}_0^C D_t^{\alpha_i} y(t) = g(t), \\ D_t^i y(0^+) = b_i, \qquad i = 0, 1, \cdots, \lfloor \alpha_0 \rfloor - 1, \end{cases}$$

where orders α_i satisfy $\alpha_0 > \alpha_1 > \cdots > \alpha_m \geq 0$ and coefficients a_i are continuous functions of t.

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Above-mentioned methods are difficult to apply for this problem. We need some new ideas.

New analytic method

A new analytic method for linear FDEs with variable coefficients emerged in the last few years from some researchers in North Korea [2],[3]:

1. Rewrite the FDE as an equivalent integral equation.

[2] M.-H. Kim, H.-C. O. Explicit representation of Green's function for linear fractional differential operator with variable coefficients, (2014).

[3] S. Pak, H. Choi, K. Sin, K. Ri. Analytical solutions of linear inhomogeneous fractional differential equation with continuous variable coefficients, (2019).

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3. Banach fixed point theorem: prove that this integral operator is a contractive mapping under the uniform norm.
4. Method of successive approximations: construct a sequence of functions which converges uniformly to the unique solution.
5. Rewrite the limit of this sequence as a uniformly convergent series of functions.

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[3] S. Pak, H. Choi, K. Sin, K. Ri. Analytical solutions of linear inhomogeneous fractional differential equation with continuous variable coefficients, (2019).

Some works and applications

[1] J.E. Restrepo, M. Ruzhansky, D. Suragan. Explicit solutions for linear variable-coefficient fractional differential equations with respect to functions. Appl. Math. Comput. 403 (2021), 126177.

[2] J.E. Restrepo, D. Suragan. Direct and inverse Cauchy problems for generalized space-time fractional differential equations. Adv. Differential Equations, 26(7–8), (2021), 305–339.

[3] A. Fernandez, J.E. Restrepo, D. Suragan. Linear differential equations with variable coefficients and Mittag-Leffler kernels. Alexandria Eng. J. 61 (2022), pp. 4757–4763.

[4] A. Fernandez, J.E. Restrepo, D. Suragan. Prabhakar-type linear differential equations with variable coefficients. Differ. Integ. Equ. 35 (2022), pp. 581–610.

[5] A. Fernandez, J.E. Restrepo, D. Suragan. On linear fractional differential equations with variable coefficients. Appl. Math. Comput. 432 (2022), 127370.

[6] J.E. Restrepo, M. Ruzhasky, D. Suragan. Generalized fractional Dirac type operators. Fract. Calc. Appl. Anal. 26, 2720–2756 (2023).

Some works and applications

[1] J.E. Restrepo, M. Ruzhansky, D. Suragan. Explicit solutions for linear variable-coefficient fractional differential equations with respect to functions. Appl. Math. Comput. 403 (2021), 126177.

[2] J.E. Restrepo, D. Suragan. Direct and inverse Cauchy problems for generalized space-time fractional differential equations. Adv. Differential Equations, 26(7–8), (2021), 305–339.

[3] A. Fernandez, J.E. Restrepo, D. Suragan. Linear differential equations with variable coefficients and Mittag-Leffler kernels. Alexandria Eng. J. 61 (2022), pp. 4757–4763.

[4] A. Fernandez, J.E. Restrepo, D. Suragan. Prabhakar-type linear differential equations with variable coefficients. Differ. Integ. Equ. 35 (2022), pp. 581–610.

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Let us briefly check how the method works !

1. Equivalent integral equation

Let us consider the FDE with homogeneous initial conditions:

$$\begin{cases} {}_0^C D_t^{\alpha_0} y(t) + \sum_{i=1}^{m} a_i(t)\, {}_0^C D_t^{\alpha_i} y(t) = g(t), \qquad t \in [0, L]; \\ D_t^i y(0^+) = 0, \qquad i = 0, 1, \cdots, \lfloor \alpha_0 \rfloor - 1. \end{cases}$$

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This is equivalent to the following integral equation (IE):

$$z(t) + \sum_{i=1}^{m} a_i(t)\, {}_{0}^{RL}I_t^{\alpha_0 - \alpha_i} z(t) = g(t),$$

via the substitution

$$z(t) = {}_{0}^{C}D_t^{\alpha_0} y(t) \qquad \longleftrightarrow \qquad y(t) = {}_{0}^{RL}I_t^{\alpha_0} z(t).$$

2. Fixed point of integral operator

The FDE is equivalent to the IE, which is in turn equivalent to $z(t)$ being a fixed point of the integral operator T defined by

$$T(z)(t) := g(t) - \sum_{i=1}^{m} a_i(t)\, {}_{0}^{RL}I_t^{\alpha_0-\alpha_i} z(t).$$

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$$T(z)(t) := g(t) - \sum_{i=1}^{m} a_i(t)\, {}^{RL}_{\ 0}I_t^{\alpha_0-\alpha_i} z(t).$$

This is a well-defined mapping from the space $C[0,L]$ (continuous functions) into itself. Equip this space with the norm $\|\cdot\|_k$ where

$$\|z\|_k = \sup_{t\in[0,L]} e^{-kt}\big|z(t)\big|,$$

where $k \in \mathbb{R}^+$ will be fixed later. This is equivalent to the uniform norm $\|\cdot\|_\infty$ on the same space.

3. Contractive mapping

Use the fact ${}^{RL}_{0}I_t^{\alpha_0-\alpha_i}e^{kt} \leq k^{-(\alpha_0-\alpha_i)}e^{kt}$ to show:

$$\begin{aligned}
\left|T(z_1)(t) - T(z_2)(t)\right| &\leq \sum_{i=1}^{m} \|a_i\|_\infty \, {}^{RL}_{0}I_t^{\alpha_0-\alpha_i}\left|z_1(t) - z_2(t)\right| \\
&\leq \|z_1 - z_2\|_k \sum_{i=1}^{m} \|a_i\|_\infty \, {}^{RL}_{0}I_t^{\alpha_0-\alpha_i} e^{kt} \\
&\leq \|z_1 - z_2\|_k \sum_{i=1}^{m} \|a_i\|_\infty \frac{e^{kt}}{k^{\alpha_0-\alpha_i}} \\
\Rightarrow \left\|Tz_1 - Tz_2\right\|_k &\leq \|z_1 - z_2\|_k \sum_{i=1}^{m} \frac{\|a_i\|_\infty}{k^{\alpha_0-\alpha_i}}.
\end{aligned}$$

For k large enough, T is a contractive mapping on $(C[0,L], \|\cdot\|_k)$, so it has a unique fixed point, i.e. a unique solution to FDE.

4. Sequence of functions

Start with a given function $z_0(t) = g(t)$ in $C[0, L]$ and apply T repeatedly to get a sequence which must converge in $\|\cdot\|_k$ (hence in $\|\cdot\|_\infty$) to the unique fixed point:

$$z_0(t) = g(t),$$

$$z_1(t) = Tz_0(t) = g(t) - \sum_{i=1}^{m} a_i(t)\, {}^{RL}_{\ 0}I_t^{\alpha_0-\alpha_i} g(t),$$

$$z_2(t) = Tz_1(t) = \sum_{k=0}^{2} (-1)^k \left[\sum_{i=1}^{m} a_i(t)\, {}^{RL}_{\ 0}I_t^{\alpha_0-\alpha_i}\right]^k g(t),$$

$$z_3(t) = Tz_2(t) = \sum_{k=0}^{3} (-1)^k \left[\sum_{i=1}^{m} a_i(t)\, {}^{RL}_{\ 0}I_t^{\alpha_0-\alpha_i}\right]^k g(t),$$

$$\dots$$

5. Uniformly convergent series

The above sequence of functions is uniformly convergent, which means the following infinite series is uniformly convergent and provides the solution $z(t)$ to the integral equation:

$$z(t)=\sum_{k=0}^{\infty}(-1)^k\left[\sum_{i=1}^{m}a_i(t)\,{}_{0}^{RL}I_t^{\alpha_0-\alpha_i}\right]^k g(t).$$

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Given the correspondence $z(t) = {}^{C}_{0}D^{\alpha_0}_{t} y(t)$, $y(t) = {}^{RL}_{0}I^{\alpha_0}_{t} z(t)$, we have the following uniformly convergent series for the unique solution $y(t)$ to the FDE:

$$y(t) = \sum_{k=0}^{\infty} (-1)^k \, {}^{RL}_{0}I^{\alpha_0}_{t} \left[\sum_{i=1}^{m} a_i(t) \, {}^{RL}_{0}I^{\alpha_0-\alpha_i}_{t} \right]^k g(t).$$

Special case: Constant-coefficients

It's easy since fractional integrals are linear and have a semigroup property:

$$
\begin{aligned}
y(t) &= \sum_{k=0}^{\infty} (-1)^k \, {}^{RL}_{0}I_t^{\alpha_0} \left[\sum_{i=1}^{m} a_i \, {}^{RL}_{0}I_t^{\alpha_0-\alpha_i} \right]^k g(t) \\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \frac{(n_1+\cdots+n_m)!}{n_1!\cdots n_m!} (-a_1)^{n_1} \cdots (-a_m)^{n_m} \\
&\qquad\qquad \times \, {}^{RL}_{0}I_t^{\alpha_0+n_1(\alpha_0-\alpha_1)+\cdots+n_m(\alpha_0-\alpha_m)} g(t) \\
&= \frac{1}{\Gamma(\alpha_0+1)} \int_0^t (t-s)^{\alpha_0-1} \times \\
&\qquad \times E_{(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_m),\alpha_0}\left(-a_1(t-s)^{\alpha_0-\alpha_1}, \cdots, -a_m(t-s)^{\alpha_0-\alpha_m}\right) g(s)\,\mathrm{d}s.
\end{aligned}
$$

Consistent with previous results (e.g. Luchko-Gorenflo).

Example in the theory of voltammetry

Let us consider the following fractional differential equation, which emerges from the theory of voltammetry at expanding electrodes [4]:

$$ {}^{RL}_{\ 0}D^{1/2}x(t) + t^{\beta-1/2}x(t) = t^{-1/2}, \qquad 0 < t \leqslant T < (\beta\sqrt{\pi})^{1/\beta}, $$
$$ {}^{RL}_{\ 0}I^{1/2}x(0) = b, $$

where $b \in \mathbb{R}^+$ and $0 < \beta \leqslant 1/2$.

[4] See page 159 of the book: K.B. Oldham, J. Spanier. The Fractional Calculus. Academic Press, New York, 1974.

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{}^{RL}_{0}I^{1/2}x(0) = b,
\end{aligned}
$$

where $b \in \mathbb{R}^+$ and $0 < \beta \leqslant 1/2$. Then the solution is given by

$$
x(t) = \frac{b}{\sqrt{\pi t}} + \sum_{k=0}^{\infty}(-1)^k \, {}^{RL}_{0}I^{1/2}\left(t^{\beta-1/2} \, {}^{RL}_{0}I^{1/2}\right)^k \left(t^{-1/2} - \frac{b}{\sqrt{\pi}}t^{\beta-1}\right).
$$

[4] See page 159 of the book: K.B. Oldham, J. Spanier. The Fractional Calculus. Academic Press, New York, 1974.

Rewritten the solution

Using the Kilbas–Saigo function [5] $E_{\alpha,m,\ell}(z) = \sum_{k=0}^{+\infty} c_k z^k$, with

$$c_0 = 1, \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma(\alpha[jm+l]+1)}{\Gamma(\alpha[jm+l+1]+\lambda+1)}, \quad k = 1, 2, \ldots,$$

one can prove that

$$\sum_{k=0}^{\infty} (-1)^k \left(t^{\alpha}\, {}^{RL}_{0}I^{\lambda} \right)^k (t^{\beta}) = t^{\beta} E_{\lambda, 1+\alpha/\lambda, \beta/\lambda} \left(-t^{\lambda+\alpha} \right)$$

whenever $\lambda > 0$ and $1 + \frac{\alpha}{\lambda} > 0$ and $j(\alpha+\lambda)+\beta \notin \mathbb{Z}^-$ for $j \in \mathbb{Z}_0^+$.
Therefore, our solution function is:

$$x(t) = \frac{b}{\sqrt{\pi t}} + {}^{RL}_{0}I^{1/2} \left(t^{-1/2} E_{1/2, 2\beta, -1} \left(-t^{\beta} \right) - \frac{b}{\sqrt{\pi}} t^{\beta-1} E_{1/2, 2\beta, 2\beta-2} \left(-t^{\beta} \right) \right).$$

[5] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. Theory and Applications of Fractional Differential Equations. Elsevier Science B.V., Amsterdam, 2006.

Fractional Leibniz rule and a more direct formula

We have fractional integrals within fractional integrals, also with function multipliers in between. To simplify this, we need the **fractional Leibniz rule**:

$$ {}^{RL}_{0}I^{\alpha}_{t}\big(f(t)g(t)\big) = \sum_{n=0}^{\infty} \binom{-\alpha}{n} {}^{RL}_{0}I^{\alpha+n}_{t} f(t) \cdot D^{n}_{t} g(t) $$

[6] A. Fernandez, J.E. Restrepo, D. Suragan. A new representation for the solutions of fractional differential equations with variable coefficients. Medit. J. Math. 20 (2023), 27.

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This gives a formula for the solution $y(t)$ which is simply a series of fractional integrals, no multiple iteration of fractional operators [6]:

$$ y(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (-1)^k \binom{-\beta}{n} \, {}^{RL}_{0}I^{k\beta+n}_{t} g(t) \times \left[\sum_{i_1+i_2+\dots+i_k=n} \frac{n!}{i_1! \dots i_k!} a^{(i_1)}(t) \dots a^{(i_k)}(t) \times \frac{(-\beta-n)!(-2\beta-i_1)!(-3\beta-i_1-i_2)! \dots (-k\beta-i_1-\dots-i_{k-1})!}{(-\beta-i_1)!(-2\beta-i_1-i_2)! \dots (-(k-1)\beta-i_1-\dots-i_{k-1})!(-k\beta-n)!} \right]. \quad (1) $$

[6] A. Fernandez, J.E. Restrepo, D. Suragan. A new representation for the solutions of fractional differential equations with variable coefficients. Medit. J. Math. 20 (2023), 27.

Conclusions

We consider differential equations with Caputo (also other) fractional derivatives and continuous (integrable) variable coefficients.

- Banach fixed point theorem $\rightarrow$ existence and uniqueness.
- Successive approximations method $\rightarrow$ explicit series solution.
- Fractional Leibniz rule $\rightarrow$ another form of the solution.

Compared to the first solution, why is the new form (1) better?

- It's much more complicated (huge formula) **but**
- No need for arbitrarily many nested fractional operators.
- Numerically easier to calculate, just infinite series of fractional integrals, coefficients are finite combinations of the $a^{(i)}(t)$.

Thank You!